LECTURE-02 (Equations of second order)

Change of Dependent variable

Method 2:→

Consider the linear equation of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{——— (1)}$$

Let $y = uv$, where $u = u(x)$ & $v = v(x)$.

$$\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\& \ \frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

putting these values in (1), we have

$$\Rightarrow u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} + \frac{1}{u}\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = \frac{R}{u} \quad \text{— (2)}$$

In Previous method, $y = u$ is known particular solution of (1) with $R = 0$. But suppose such a solution is not known. Then we choose the function u s.t. the coefficient of $\frac{dv}{dx}$ in (2) is zero.

$$\therefore \frac{2}{u}\frac{du}{dx} + P = 0$$

$$\therefore u = e^{-\frac{1}{2}\int P\,dx} \quad \text{——— (3)}$$

With this value of u, the given equation reduces to the form

$$\frac{d^2 v}{dx^2} + Q_1 v = R_1$$

Where $Q_1 = \frac{1}{u}\left\{\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right\}$ & $R_1 = \frac{R}{u}$

Now, $\because u = e^{-\frac{1}{2}\int P dx}$

$$\therefore R_1 = \frac{R}{u} = R\, e^{\frac{1}{2}\int P dx}$$

$$\& \ \frac{du}{dx} = -\frac{1}{2} P e^{-\frac{1}{2}\int P dx} = -\frac{1}{2} P u$$

$$\therefore \frac{d^2u}{dx^2} = -\frac{1}{2}\left[P\frac{du}{dx} + u\frac{dP}{dx}\right]$$

$$= -\frac{1}{2}\left[P\left(\frac{1}{2}Pu\right) + u\frac{dP}{dx}\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{2}P^2 u + u\frac{dP}{dx}\right] = -\frac{1}{2}u\left(\frac{dP}{dx} - \frac{1}{2}P^2\right)$$

$$\therefore Q_1 = \frac{1}{u}\left\{\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right\}$$

$$= \frac{1}{u}\left\{-\frac{1}{2}u\left(\frac{dP}{dx} - \frac{1}{2}P^2\right) + P\left(-\frac{1}{2}Pu\right) + Qu\right\}$$

$$= \frac{1}{u}\cdot u\left\{-\frac{1}{2}\cdot\frac{dP}{dx} + \frac{1}{4}P^2 - \frac{1}{2}P^2 + Q\right\}$$

$$\therefore Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$$

Then the transformed equation ② becomes

$$\frac{d^2v}{dx^2} + Q_1 v = R \cdot e^{\frac{1}{2}\int P\,dx} \qquad \text{③}$$

Which can be solved using different method.

<u>Example ①</u> Solve, $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(1 + \frac{2}{x^2}\right)y = xe^x$

<u>Solution:-</u> Where $P = -\frac{2}{x}$, $Q = 1 + \frac{2}{x^2}$ & $R = xe^x$

$$\therefore u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\cdot\int -\frac{2}{x}\,dx} = e^{\log x} = x$$

$$\text{And } R_1 = Re^{\frac{1}{2}\int P\,dx} = \frac{R}{u} = \frac{xe^x}{x} = e^x$$

$$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$$

$$= 1 + \frac{2}{x^2} - \frac{1}{2}\left(\frac{2}{x^2}\right) - \frac{1}{4}\left(\frac{4}{x^2}\right) = 1$$

$\therefore$ Under this transformation $y = v \cdot u = v \cdot x$,

The given equation reduced to

$$\frac{d^2v}{dx^2} + Q_1 v = R_1$$

i,e, $$\frac{d^2v}{dx^2} + v = e^x$$

This is a linear equation in v with constant coefficients.

$\therefore$ A·E $\equiv D^2 + 1 = 0 \Rightarrow D = \pm i$

$\therefore$ C.F. $= A\cos x + B\sin x$

& $P.I. = \frac{1}{D^2+1} e^x = \frac{1}{2} e^x$

$\therefore v = A\cos x + B\sin x + \frac{1}{2} e^x$

$\therefore$ The complete primitive of the given equation is

$$y = vx$$

$$\therefore y = Ax\cos x + Bx\sin x + \frac{1}{2} x e^x$$

Ans.

Example ② Solve

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 4x^2 y = x e^{x^2}$$

Example ③ :- Solve

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = 0$$